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An Addition Formula For Green's Functions

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1. INTRODUCTION

One of the most general approaches to solving boundary value problems for linear differential equations consists of expressing the solution in terms of an auxiliary function called the Green's function. The Green's function is usually defined to be the solution of the adjoint problem with a delta function as a non-homogeneous term. The particular boundary conditions affect the formulation of the adjoint problem and thus, the Green's function as well. The Green's function method can be applied to elliptic, parabolic, and hyperbolic linear partial differential equations as well as linear ordinary differential equations.

This paper presents a derivation of a formula that gives a form of the Green's function of a separable elliptic partial differential equation in terms of the Green's functions for two simpler equations. Although other methods exist for attacking separable problems, such as separation of variables and transform techniques, this new method presents an additional approach which may on occassion be more efficient. Furthermore, while the original elliptic equation is assumed separable, the formula may work with some non-separable boundary conditions as well. An example of this phenomenon is seen in section 4. Of the two simpler equations involved in the method, one is elliptic and the other hyperbolic. Because of the presence of the hyperbolic equation and for the sake of completeness, we shall, in section 2, briefly review the definition of the Green's function for a Cauchy problem and its relationship to the Riemann function. Then, the Riemann function, which is known in some special cases, can be used to find the Green's function in these cases. In section 3, the formula will be formally derived, applied to the reduced wave equation, and be further justified. In section 4, we shall apply the formula to derive what we believe to be a new Green's function.

Before proceeding however, it is interesting to note that although we are primarily interested in Green's functions for elliptic problems, the motivation

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for this work came from recent papers on hyperbolic problems. A second-order linear hyperbolic equation in two independent variables may be solved using the Riemann function. In 1958, Copson [Ref. 1] formally derived a technique giving a possible form for the Riemann function (which Copson called the Riemann–Green function), assuming the variables are separable. The technique utilized integral transforms which generalized Riemann's own use of Fourier cosine transforms. In 1964, Mackie [Ref. 2] investigated similar problems and presented a close relationship between the Riemann function and the Green's function for the Cauchy problem. Mackie then applied Fourier–Bessel transforms to find the Green's function for Riemann's original example and utilized the relationship between Riemann and Green's functions to find the Riemann functions. Recently, Papadakis and Wood [Ref. 3] developed an addition formula that gives the Riemann function of a separable equation in terms of the Riemann functions of two simpler equations. In deriving the formula, they employed a refinement of Copson's integral transform method to find Green's functions and then utilized Mackie's relationship to find the Riemann function. In this paper, the technique is further developed by applying it to Green's functions for elliptic partial differential equations.

2. Preliminaries

a. *The Green's Function for Elliptic Equations*

Consider the linear elliptic second-order partial differential equation in two variables

$$E[u] = u_{xx} + u_{yy} + 2a(x, y)\, u_x - 2b(x, y)\, u_y + c(x, y)\, u(x, y) = f(x, y) \quad (2.1)$$

with suitable boundary conditions on a closed curve Γ (e.g., $u = 0$ on Γ). The Green's function for (2.1), $G(x, y; X, Y)$ is defined as the solution to the adjoint equation

$$E^*[G] = G_{xx} + G_{yy} - 2(aG)_x + 2(bG)_y + cG = \delta(x - X)\, \delta(y - Y) \quad (2.2)$$

with suitable (adjoint) conditions on the boundary Γ (e.g., $G = 0$ on Γ). If D is the interior of Γ, the solution of (2.1) can be written in the form

$$u(X, Y) = \int_D\int G(x, y; X, Y) f(x, y)\, dx\, dy + \int_\Gamma B[u, G] \quad (2.3)$$

where $B[u, G]$ is known on Γ (e.g., $B[u, G] = 0$ on Γ).

b. *The Green's function for Hyperbolic Equations*

The concept of the Green's function is also meaningful for the Cauchy problem

$$H[u] = u_{xx} - u_{yy} + 2a(x, y)\, u_x - 2b(x, y)\, u_y + c(x, y)\, u(x, y) = f(x, y) \quad (2.4)$$

with initial values u and u_y given on the line $y = y_0$. As in the elliptic case, the Green's function $G(x, y; X, Y)$ is the solution of

$$H^*[G] = G_{xx} - G_{yy} - 2(aG)_x + 2(bG)_y + cG = \delta(x - X)\,\delta(y - Y) \tag{2.5}$$

subject to the adjoint boundary conditions

$$G = \frac{\partial G}{\partial n} = 0 \quad \text{on} \quad \Gamma \tag{2.6}$$

where $\partial G/\partial n$ is the (outward) normal derivative of G on Γ, and Γ is a non-charactzristic curve (i.e., the absolute value of the slope of Γ is always less than 1) such that Γ and a segment, $\mathscr{L}$, of the line $y = y_0$ form a closed curve, as in Fig. 1. From (2.5), (2.6), and the amount of freedom we have in choosing Γ, it

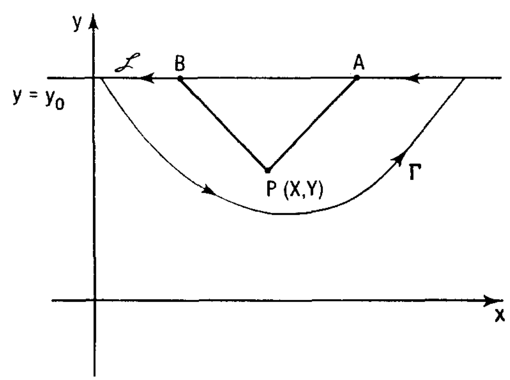

FIGURE 1

can be seen that $G = 0$ in the portion of the interior of $\Gamma + \mathscr{L}$ outside the triangle Δ, with vertices at points P, A, and B. Now the solution u of (2.4) is given by

$$u(X, Y) = \iint_\Delta G(x, y; X, Y) f(x, y)\, dx\, dy - \int_{\mathscr{L}} (u_y G - u G_y + 2buG)\, dx. \tag{2.7}$$

c. *The Riemann Function*

For the Cauchy problem described above, the Riemann function $R(x, y; X, Y)$ is the solution of

$$H^*[R] = 0 \tag{2.8a}$$

satisfying

$$R_x + R_y = (a + b)\, R \quad \text{on} \quad y - Y = x - X \tag{2.8b}$$

$$R_x - R_y = (a - b)\, R \quad \text{on} \quad y - Y = -(x - X) \tag{2.8c}$$

$$R(X, Y; X, Y) = 1. \tag{2.8d}$$

Now the solution of (2.4) is given by

$$u(X, Y) = -\tfrac{1}{2}\iint_{\Delta} R(x, y; X, Y) f(x, y)\, dx\, dy + \tfrac{1}{2}\int_{A}^{B} (u_y R - uR_y + 2buR)\, dx$$
$$+ \tfrac{1}{2}[u(A)\, R(A; P) + u(B)\, R(B; P)]. \tag{2.9}$$

If the initial values of u and u_y are homogeneous (i.e., identically 0) along $y = y_0$, a comparison of (2.7) and (2.9) would lead to

$$G(x, y; X, Y) = -\tfrac{1}{2}R(x, y; X, Y) \tag{2.10}$$

inside the triangle Δ. If the initial data along $y = y_0$ is nonhomogeneous, the contour integral in (2.7) requires integration of terms including G and G_y along lines across which G itself is discontinuous. This generates terms analogous to those inside the brackets in (2.9). Thus, (2.10) holds for arbitrary initial data. (Except for a difference in notation between a Green's function and an adjoint Green's function, the analysis leading to (2.10) is due to Mackie [Ref. 2].)

3. The Addition Formula

a. *Formal Derivation*

Let us consider the separable elliptic equation

$$L[u] = u_{xx} + u_{yy} + [c_1(x) + c_2(y)]\, u(x, y) = f(x, y) \tag{3.1}$$

and attempt to find the "free-space" Green's function (i.e., assuming $c_1(x)$ and $c_2(y)$ are analytic, find a Green's function over the entire (x, y) plane). Since L is self-adjoint, $G(x, y; X, Y)$ must satisfy

$$L^*[G] \equiv L[G] = \delta(x - X)\, \delta(y - Y). \tag{3.2}$$

We shall attempt to find an expression for G in terms of the Green's function G_1 satisfying the hyperbolic equation

$$G_{1xx} - G_{1yy} + c_1(x)\, G_1 = \delta(x - X)\, \delta(y - Y) \tag{3.3}$$

and the Green's function G_2 satisfying the elliptic equation

$$G_{2xx} + G_{2yy} + c_2(y)\, G_2 = \delta(x - X)\, \delta(y - Y). \tag{3.4}$$

It has been shown by Papadakis and Wood [Ref. 3] that

$$R_1(x, y; X, Y) \equiv R_1(x, y - Y; X, 0) \equiv R_1(x, Y - y; X, 0). \tag{3.5}$$

Recalling (2.10) we can say that G_1 is also an even function of $y - Y$ wherever it is defined for both $+(y - Y)$ and $-(y - Y)$. We should like to make a similar claim for G_2. If we assume a condition at ∞, call it a "radiation condition", insuring the uniqueness of G_2, then we can claim that

$$G_2(x, y; X, Y) \equiv G_2(x - X, y; 0, Y) \equiv G_2(X - x, y; 0, Y). \tag{3.6}$$

Clearly, any form of (3.6) satisfies (3.4). Thus, if our "radiation condition" is also satisfied by any form of (3.6), the claim is justified by the uniqueness of G_2. Since we are proceeding formally at this stage, let us accept these hypotheses and (3.6) as established, with this final observation. If $c_2(y) \equiv k^2$ (the case with which we will be primarily concerned in this paper) and the well-known Sommerfeld radiation condition [Ref. 4] is assumed, then G_2 is unique and an even function of $x - X$ as desired.

Now, let us proceed by separating variables in the homogeneous form of (3.2), leading to a pair of ordinary differential equations, one of which is

$$\theta''(x) + [c_1(x) + \lambda^2]\,\theta(x) = 0 \tag{3.7}$$

where λ^2 is the separation constant. Assume there exists a solution $\theta(x, \lambda)$ of (3.7) that defines a transform

$$f(\lambda) = \int \theta(x, \lambda) F(x)\, dx.$$

Let $\hat{\theta}(x, \lambda)$ be the inverse transform so that

$$F(x) = \int \hat{\theta}(x, \lambda) f(\lambda)\, d\lambda.$$

Applying the transform to (3.2), integrating by parts twice (ignoring terms evaluated at the endpoints of the integral), and recalling that $\theta(x, \lambda)$ is a solution of (3.7) yields

$$g_{yy} + [c_2(y) - \lambda^2]\, g = \delta(y - Y) \tag{3.9}$$

where g is defined by

$$g(y; X, Y, \lambda)\, \theta(X, \lambda) \equiv \int \theta(x, \lambda)\, G(x, y; X, Y)\, dx.$$

Utilizing the inverse transform gives

$$G(x, y; X, Y) = \int \hat{\theta}(x, \lambda)\, \theta(X, \lambda)\, g(y; X, Y, \lambda)\, d\lambda. \tag{3.10}$$

Now let us apply the same procedure to (3.3). Separating variables leads to (3.7) again and therefore, the same solution $\theta(x, \lambda)$. Transforming (3.3) yields

$$g_{1yy} + \lambda^2 g_1 = -\delta(y - Y) \tag{3.11}$$

where

$$g_1(y - Y; X, \lambda)\,\theta(X, \lambda) \equiv \int \theta(x, \lambda)\, G_1(x, y - Y; X, 0)\, dx. \tag{3.12}$$

Recalling boundary condition (2.6) and Fig. 1, (3.11) can be solved uniquely with the initial conditions $g_1 \to 0$ and $g_{1y} \to 0$ as $y \to -\infty$:

$$g_1(y - Y; X, \lambda) = -\frac{\sin \lambda(y - Y)}{\lambda} H(y - Y)$$

where H is the Heaviside step function. Inverse transforming (3.12) yields

$$G_1(x, y - Y; X, 0) = -H(y - Y) \int \hat{\theta}(x, \lambda)\, \theta(X, \lambda) \frac{\sin \lambda(y - Y)}{\lambda}\, d\lambda. \tag{3.13}$$

For the last time, separating variables in (3.4) leads to the equation

$$\phi''(x) + \lambda^2 \phi(x) = 0$$

with a solution of the form $e^{-i\lambda x}$, yielding a Fourier transform. As before, we transform (3.4) and find

$$g_{2yy} + [c_2(y) - \lambda^2]\, g_2 = \delta(y - Y) \tag{3.14}$$

where

$$g_2(y; X, Y, \lambda)\, e^{-i\lambda X} \equiv \int_{-\infty}^{\infty} e^{-i\lambda x} G_2(x - X, y; 0, Y)\, dx. \tag{3.15}$$

Recalling from (3.6) that G_2 is an even function of $x - X$, it is easily deduced that g_2 is an even function of λ and independent of X. Also comparing (3.14) to (3.9) leads to the conclusion that $g_2 \equiv g$ and therefore

$$g(y; X, Y, \lambda) \equiv g(y, Y; \lambda) \equiv g(y, Y; -\lambda).$$

Now, inverse transforming (3.15) and using the fact that G_2 and g are even

$$G_2(x - X, y; 0, Y) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda(x - X)\, g(y, Y; \lambda)\, d\lambda. \tag{3.16}$$

Let us formally assume that the integrals in (3.10) and (3.13) also have limits from 0 to ∞. If, in (3.13), we denote $y - Y$ by t and differentiate with respect to t we get

$$G_{1t}(x, t; X, 0) = -\int_0^{\infty} \hat{\theta}(x, \lambda)\, \theta(X, \lambda) \cos \lambda t\, d\lambda. \tag{3.17}$$

Rewriting (3.16) with t replacing $x - X$

$$G_2(t, y; 0, Y) = \frac{1}{\pi} \int_0^{\infty} g(y, Y; \lambda) \cos \lambda t\, d\lambda. \tag{3.18}$$

An application of Parseval's identity to (3.17) and (3.18) yields

$$\int_0^\infty G_{1t}(x, t; X, 0)\, G_2(t, y; 0, Y)\, dt = -\tfrac{1}{2} \int_0^\infty \hat{\theta}(x, \lambda)\, \theta(X, \lambda)\, g(y, Y; \lambda)\, d\lambda$$
$$= -\tfrac{1}{2} G(x, y; X, Y)$$

the last equality coming from (3.10), with the assumed limits of integration from 0 to ∞. Thus we hope to find the Green's function of (3.1) in the form

$$G(x, y; X, Y) = -2 \int_0^\infty G_{1t}(x, t; X, 0)\, G_2(t, y; 0, Y)\, dt. \tag{3.19}$$

Equation (3.19) is the general form that we shall use in our search for Green's functions. In order to get a more specific formula, one must investigate the behavior of G_1 and G_2 and the effects of boundaries and boundary conditions on G. For the time being, we shall continue on with our assumptions that $c_1(x)$ and $c_2(y)$ are well-behaved functions and look for a "free-space" formula. In section 4, we shall analyze (3.19) under different conditions.

Now, recalling our discussion of Green's functions and Riemann functions for hyperbolic equations, and particularly Fig. 1 and Eq. (2.10)

$$\begin{aligned} G_1(x, y; X, Y) &= 0 && \text{for } y - Y < | x - X | \\ &= -\tfrac{1}{2} R_1(x, y; X, Y) && \text{for } y - Y \geqslant | x - X | \end{aligned}$$

where the Riemann function, R_1, is defined by (2.8). Therefore, we can write

$$G_1(x, t; X, 0) = -\tfrac{1}{2} R_1(x, t; X, 0)\, H(t - | x - X |)$$

where H is once again the Heaviside step function. Differentiating G_1 with respect to t:

$$\begin{aligned} &G_{1t}(x, t; X, 0) \\ &\quad = -\tfrac{1}{2} R_1(x, t; X, 0)\, \delta(t - | x - X |) - \tfrac{1}{2} R_{1t}(x, t; X, 0)\, H(t - | x - X |). \end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19):

$$\begin{aligned} G(x, y; X, Y) &= \int_0^\infty \{R_1 \delta + R_{1t} H\}\, G_2\, dt \\ &= R_1(x, | x - X |; X, 0)\, G_2(| x - X |, y; 0, Y) \\ &\quad + \int_{|x-X|}^\infty R_{1t}(x, t; X, 0)\, G_2(t, y; 0, Y)\, dt. \end{aligned} \tag{3.21}$$

Integrating (2.8b and c) with $a = b = 0$, we get

$$R_1(x, \mid x - X \mid; X, 0) = 1.$$

From (3.6)

$$G_2(\mid x - X \mid, y; 0, Y) = G_2(x - X, y; 0, Y).$$

Lastly, R_1 and G_2, as they appear in the integral in (3.21), are even functions of t so that the integrand (with its t-derivative term) is odd and we can drop the absolute value from the lower limit of integration.

Recalling (2.10) we can now write (3.21) as

$$G(X, y; X, Y) = G_2(x - X, y; 0, Y) - 2 \int_{x-X}^{\infty} G_{1t}(x, t; X, 0)\, G_2(t, y; 0, Y)\, dt. \tag{3.22}$$

Integrating by parts and assuming that the term $G_1 G_2$ evaluated at $t = \infty$ is zero, yields the alternate form

$$G(x, y; X, Y) = 2 \int_{x-X}^{\infty} G_1(x, t; X, 0)\, G_{2t}(t, y; 0, Y)\, dt. \tag{3.23}$$

Equations (3.22) and (3.23) are the relationships between the Green's functions G, G_1, and G_2 satisfying (3.2), (3.3), and (3.4) which we wished to derive from our general form (3.19). Of course, all our work so far is entirely formal and thus, justification is required. Although we do not intend to produce a rigorous proof of the validity of the formulas, we shall verify that they satisfy some of the properties of Green's functions and further justify some of our assumptions. But before continuing in this endeavor, let us convince ourselves that it is worth the effort by testing the formulas on a problem for which we know the correct answer.

b. *A Test Problem*

In this section, we shall utilize (3.23) to find the Green's function for (3.1) with $c_1(x) = -a^2$ and $c_2(y) = b^2$, where a and b are real, positive constants satisfying

$$k^2 = b^2 - a^2 > 0. \tag{3.24}$$

Notice that we have broken up the positive constant k^2 as the difference of two positive constants rather than the sum. We shall comment on this choice later in the section.

Now we can rewrite (3.2), (3.3), and (3.4), the equations for G, G_1, and G_2 respectively as

$$G_{xx} + G_{yy} + k^2 G = \delta(x - X)\,\delta(y - Y), \tag{3.25}$$

$$G_{1xx} - G_{1yy} - a^2 G_1 = \delta(x - X)\,\delta(y - Y), \tag{3.26}$$

and

$$G_{2xx} + G_{2yy} + b^2 G_2 = \delta(x - X)\,\delta(y - Y). \tag{3.27}$$

The solutions of (3.26) and (3.27) are known to be

$$G_1(x, y; X, Y) = -\tfrac{1}{2} J_0(a[(y - Y)^2 - (x - X)^2]^{1/2}) \qquad \text{for } y - Y \geqslant |x - X|$$
$$= 0 \qquad \text{for } y - Y < |x - X| \tag{3.28}$$

and

$$G_2(x, y; X, Y) = \frac{1}{4i} H_0^{(1)}(b[(x - X)^2 + (y - Y)^2]^{1/2}). \tag{3.29}$$

Substituting (3.28) and (3.29) into (3.23) and then performing the required differentiation yields

$$G(x, y; X, Y)$$
$$= \frac{-1}{4i} \int_{x-X}^{\infty} J_0(a[t^2 - (x - X)^2]^{1/2}) \frac{\partial}{\partial t} H_0^{(1)}(b[t^2 + (y - Y)^2]^{1/2})\, dt$$
$$= \frac{1}{4i} \int_{x-X}^{\infty} J_0(a[t^2 - (x - X)^2]^{1/2}) H_1^{(1)}(b[t^2 + (y - Y)^2]^{1/2}) \frac{bt\, dt}{[t^2 + (y - Y)^2]^{1/2}}\,.$$

Changing the integration variable from t to

$$s = [t^2 - (x - X)^2]^{1/2}$$

yields

$$G(x, y; X, Y) = \frac{1}{4i} \int_0^{\infty} J_0(as) H_1^{(1)}(b[s^2 + r^2]^{1/2}) \frac{bs\, ds}{[s^2 + r^2]^{1/2}} \tag{3.30}$$

where $r \equiv [(x - X)^2 + (y - Y)^2]^{1/2}$.
Utilizing (3.24) and the appropriate integral tables [Ref. 5, pg. 358, 19.4(3) and Ref. 6, pg. 706, 6.596(6)] yields

$$G(x, y; X, Y) = \frac{1}{4i} H_0^{(1)}(kr) \tag{3.31}$$

the well-known free-space Green's function for the reduced wave equation (3.25).

Returning to the remark following (3.24), we note that if k^2 were the sum of a^2 b^2 then, in (3.28), G_1 would consist of an I_0, a modified Bessel function, rather than a J_0. If, in (3.30), the J_0 term were replaced by I_0, the integral would diverge since I_0 increases exponentially for large arguments. Thus, even under conditions when the formula is applicable, each problem must be studied with great care and a certain amount of ingenuity may be required in order to solve the problem.

Let us note that the formula also works for the case $c_1(x) = -a^2$ and $c_2(y) = -b^2$. The analysis is almost identical with the preceding example, with the elliptic Green's function $(1/4i)\,H_0^{(1)}$ replaced by $(-1/2\pi)\,K_0$, the modified Hankel function. Reference 6 [pg. 706, 6.596(6)] provides the required integral.

c. *Further Justification of the Addition Formula*

In order to verify the addition formula (3.19) or formulas derived from it, such as (3.22) and (3.23), at least three properties of our representation of G should be checked:

(1) that the integrals involved converge;

(2) that as $(x, y) \to (X, Y)$, G possesses the appropriate (logarithmic) singularity; and

(3) that for $(x, y) \neq (X, Y)$, G satisfies the homogeneous equation.

Properties (1) and (2) should be verified for each specific problem investigated; that is, for the particular G_1 and G_2 that happen to arise. For formulas (3.22) and (3.23), property (3) can then be verified in general by direct differentiation.

Differentiating (3.23) twice with respect to x and recalling that for $(x, y) \neq (X, Y)$, $G_{1xx} = G_{1yy} - c_1 G_1$ (from (3.3))

$$G_{xx} = G_{2xx} - 2G_{2x}(G_{1x}\,|_{y-Y=x-X}) + 2\int_{x-X}^{\infty} G_{1tt}G_{2t}\,dt - c_1 G. \tag{3.32}$$

Similarly, differentiating (3.22) twice with respect to y and recalling that $G_{2xx} = -\,G_{2yy} - c_2 G_2$ (from (3.4))

$$G_{yy} = G_{2yy} + 2\int_{x-X}^{\infty} G_{1t}G_{2tt}\,dt + 2c_2\int_{x-X}^{\infty} G_{1t}G_2\,dt. \tag{3.33}$$

Adding (3.32) and (3.33)

$$\begin{aligned} G_{xx} + G_{yy} &= -c_1 G + (G_{2xx} + G_{2yy}) + 2c_2\int_{x-X}^{\infty} G_{1x}G_2\,dt \\ &\quad + 2\int_{x-X}^{\infty} (G_{1t}G_{2tt} + G_{1tt}G_{2t})\,dt - 2G_{2x}(G_{1x}\,|_{y-Y=x-X}) \\ &= -c_1 G - c_2\left(G_2 - 2\int_{x-X}^{\infty} G_{1t}G_2\,dt\right) + 2\int_{x-X}^{\infty} (G_{1t}G_{2t})_t\,dt \\ &\quad - 2G_{2x}(G_{1x}\,|_{y-Y=x-X}). \end{aligned}$$

So, using (3.22)

$$G_{xx} + G_{yy} + (c_1 + c_2)\,G = 2(G_{1t}G_{2t})\,|_{t=x-X}^{\infty} - 2G_{2x}(G_{1x}\,|_{y-Y=x-X}).$$

Recalling that we have already assumed that $G_1G_2 \to 0$ as $t \to \infty$ in order to integrate by parts let us further assume that $G_{1t}G_{2t} \to 0$ as $t \to \infty$. Then

$$G_{xx} + G_{yy} + (c_1 + c_2)\,G = -2G_{2x}(G_{1x} + G_{1y})\,|_{y-Y=x-X} = 0$$

by (2.10) and integration of (2.8b, with $a = b = 0$). Thus (3.22) and (3.23) satisfy the homogeneous equation away from the source if

$$\lim_{t\to\infty} G_1(x, t; X, 0)\, G_2(t, y; 0, Y) = 0 \tag{3.34}$$

and

$$\lim_{t\to\infty} G_{1t}(x, t; X, 0)\, G_{2t}(t, y; 0, Y) = 0. \tag{3.35}$$

These conditions were met by the Green's functions in our test problem.

Let us note that in some cases, conditions (3.34) and (3.35) may be unnecessarily restrictive. In particular (3.34) was used to integrate (3.22) by parts and thus deriving (3.23). Then we used the equivalence of (3.22) and (3.23), along with (3.35), to verify that our formula did indeed yield a solution of the homogeneous differential equation. However, it is possible for (3.22) (or (3.23), for that matter) to be valid while (3.23) (or (3.22)) is not.

Consider Laplace's equation

$$u_{xx} + u_{yy} = 0. \tag{3.36}$$

With $c_1(x)$ and $c_2(y)$ both 0, G_1 and G_2 are the Green's functions for

$$u_{xx} - u_{yy} = 0 \tag{3.37}$$

and

$$u_{xx} + u_{yy} = 0 \tag{3.38}$$

respectively.

Then,

$$\begin{aligned} G_1(x, y; X, Y) &= -\tfrac{1}{2} \qquad \text{for } y - Y \geqslant |x - X| \\ &= 0 \qquad \text{for } y - Y < |x - X| \end{aligned} \tag{3.39}$$

and, letting $r \equiv [(x - X)^2 + (y - Y)^2]^{1/2}$

$$G_2(x, y; X, Y) = \frac{1}{2\pi} \log r. \tag{3.40}$$

Substituting in (3.22) we find the well-known free-space Green's function for Laplace's equation

$$G(x, y; X, Y) = \frac{1}{2\pi} \log r \tag{3.41}$$

since $G_{1t} = 0$ inside the integral appearing in (3.22). However, (3.23) diverges for the same situation.

Now, in light of our successful general verification of property (3) and our successful applications of the formulas in the test example and Laplace's equation, let us take a heuristic look at properties (1) and (2) by placing certain restrictions on $c_1(x)$ and $c_2(y)$ and the boundary conditions associated with (3.1). For the formal derivation we assumed that c_1 and c_2 were analytic in the entire plane. Now let us modify this assumption so that rather than the whole plane, we are looking at an infinite domain with boundaries separable in x and y (e.g., the half-plane $x > 0$). Also let us assume that any singularities in c_1 and c_2 appear outside the domain of interest. Finally, assume that as x and y approach ∞, $c_1(x)$ and $c_2(y)$ approach real constant values. Witout loss of generality, it can be assumed that c_1 approaches a negative constant since we can substract a constant from c_1 and add it to c_2 and not alter the total coefficient in (3.1).

With these assumptions on c_1 and c_2 we can hope that G_1 and G_2, satisfying (3.3) and (3.4), will behave, near ∞, like J_0 and $H_0^{(1)}$ (or K_0), the Green's functions satisfying (3.26) and (3.27). From the test problem, we know that the integral (3.30) converges and thus hope that the integrals containing G_1 and G_2, (3.22) and (3.23), will also converge and that the integration by parts leading from one form to the other will be justified. If necessary, further restrictions on how fast c_1 and c_2 approach their limiting values could be made to insure asymptotic knowledge of G_1 and G_2, or perhaps entirely different types of assumptions are necessary. Let us note that in section 4, we shall, for a specific example, derive a form of (3.23) with finite upper and lower limits of integration. In this case, property (1), the convergence of the formula, is no longer a problem to establish.

We shall cope with the problem of the singularity as $(x, y) \to (X, Y)$ in much the same vein. Since we have assumed that c_1 and c_2 are well-behaved, we shall also assume that they can be treated locally as constants. With these assumptions we can hope that the general G_1 and G_2 will behave like their counterparts in the test problem and exhibit the appropriate singularity.

While these arguments indicate that properties (1) and (2) will be satisfied by certain types of interesting problems, let us recall that these properties should be checked for each individual problem. Property (3), however, has been directly verified under general conditions.

4. The Green's Function for $u_{xx} + u_{yy} + [k^2 - \{(m(m+1))/x^2\}]\, u = 0$.

In this section we shall finally apply the addition formula to a non-trivial example. To the best of our knowledge, the Green's function found in this section has not been previously presented, at least not in the particularly simple forms we shall derive.

Let us consider the equation

$$L[u] = u_{xx} + u_{yy} + \left[k^2 - \frac{m(m+1)}{x^2}\right] u = 0 \tag{4.1}$$

on the half-plane $x > 0$, where m is a positive integer. We shall attempt to find the Green's function $G(x, y; X, Y)$ for (4.1) on the half-plane $x > 0$ satisfying

$$L[G] = \delta(x - X)\,\delta(y - Y) \tag{4.2}$$

$$G = 0 \qquad \text{on} \qquad x = 0 \tag{4.2a}$$

$$G \to 0 \qquad \text{as} \qquad (x^2 + y^2)^{1/2} \to \infty \tag{4.2b}$$

Note that (4.2b) is a non-separable condition.

As (4.1) seems to suggest we shall apply our formula with $c_1(x) = -m(m+1)/x^2$ and $c_2(y) = k^2$. Thus, we must find the Green's function G_1 for

$$u_{xx} - u_{yy} - \frac{m(m+1)}{x^2}\,u = 0. \tag{4.3}$$

As we have seen previously, G_2 must satisfy (3.25); so

$$G_2(x - X, y; 0, Y) = \frac{1}{4i}\,H_0^{(1)}(kr)$$

where $r \equiv [(x - X)^2 + (y - Y)^2]^{1/2}$.

In order to avoid an unecessary digression, we shall show in Appendix 1 that

$$\begin{aligned} G_1(x, y - Y; X, 0) &= -\tfrac{1}{2}P_m(\xi) && \text{for } |x - X| < y - Y < x + X \\ &= 0 && \text{elsewhere} \end{aligned} \tag{4.4}$$

where $\xi \equiv [x^2 + X^2 - (y - Y)^2]/2xX$ and P_m is the Legendre polynomial of the 1st kind (see Fig. 2).

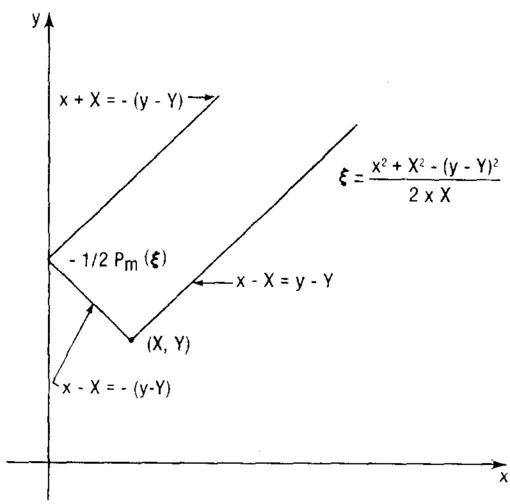

FIGURE 2

Now that G_1 and G_2 are known, we can return to our general form of the addition formula, Eq. (3.19). Then (4.4) may also be written in the form

$$G_1(x, t; X, 0) = -\tfrac{1}{2}P_m\left(\frac{x^2 + X^2 - t^2}{2xX}\right) H(t - |x - X|)\, H(x + X - t).$$

Noting that $P_m(\xi)$ is the Riemann function $R_1(x, y; X, Y)$ for (4.3), we can also write

$$G_1(x, t; X, 0) = -\tfrac{1}{2}R_1(x, t; X, 0)\, H(t - |x - X|)\, H(x + X - t). \tag{4.5}$$

Differentiating with respect to t

$$\begin{aligned} G_{1t}(x, t; X, 0) = &-\tfrac{1}{2}R_{1t}(x, t; X, 0)\, H(t - |x - X|)\, H(x + X - t) \\ &-\tfrac{1}{2}R_1(x, t; X, 0)\, \delta(t - |x - X|)\, H(x + X - t) \\ &+\tfrac{1}{2}R_1(x, t; X, 0)\, H(t - |x - X|)\, \delta(x + X - t). \end{aligned}$$

Substitution into (3.19) yields

$$\begin{aligned} G &= -2\int_0^\infty G_{1t}G_2\, dt \\ &= \int_{|x-X|}^{x+X} R_{1t}G_2\, dt + R_1|_{y-Y=|x-X|}\, H(x + X - |x - X|)\, G_2|_{y-Y=|x-X|} \\ &\quad - R_1|_{y-Y=x+X}\, H(x + X - |x - X|)\, G_2|_{y-Y=x+X}\,. \end{aligned} \tag{4.6}$$

Since x and X are both positive, the Heaviside functions are both 1. Since R_1 is the Riemann function for (4.3), $R_1 = 1$ along the characteristic line $y - Y = x - X$; a fact we could also have found by noting that $P_m|_{y-Y=|x-X|} = P_m(1) = 1$. Although we also know $P_m|_{y-Y=x+X} = P_m(-1) = (-1)^m$, we shall not use this fact at this stage of the analysis because it appears to be too specialized (i.e., cannot be deduced from properties of the Riemann function) to include in a general form of the formula. Now, since the integrations over the discontinuities of G_1 have been taken into account, we can replace R_1 by $-2G_1$ and using the now familiar argument that G_2 is an even function of t and that the integrand in (4.6) is odd, we get

$$\begin{aligned} G(x, y; X, Y) = &-2\int_{x-X}^{x+X} G_{1t}(x, t; X, 0)\, G_2(t, y; 0, Y)\, dt + G_2(x - X, y; 0, Y) \\ &+ 2G_1(x, x + X; X, 0)\, G_2(x + X, y; 0, Y). \end{aligned} \tag{4.7}$$

Equation (4.7) is the counterpart of (3.22). As a matter of fact, in this case we did not need to go all the way back to (3.19). We could have substituted our expression for G_{1t}, (4.5), into (3.22) and gotten (4.7). However, because we have not proven a theorem as to the applicability of the formula, we gain more confidence by reanalyzing (3.19) than by performing blind substitutions. Now, integrating (4.7) by parts yields the counterpart of (3.23)

$$G(x, y; X, Y) = 2 \int_{x-X}^{x+X} G_1(x, t; X, 0)\, G_{2t}(t, y; 0, Y)\, dt. \tag{4.8}$$

Direct differentiation of (4.7) and (4.8), in a manner analogous to that performed in section 3c, verifies that G is indeed a solution of (4.1) for $(x, y) \neq (X, Y)$. Let us note that because we have finite limits of integration, we no longer need conditions (3.34) and (3.35).

Substituting our known forms of G_1 and G_2 into (4.7) and (4.8), with the added notational convenience $\bar{r} \equiv [(x + X)^2 + (y - Y)^2]^{1/2}$, yields

$$\begin{aligned} G(x, y; X, Y) = \frac{1}{4i} H_0^{(1)}(kr) - (-1)^m H_0^{(1)}(k\bar{r}) \\ + \int_{x-X}^{x+X} \frac{\partial Pm}{\partial t} \left(\frac{x^2 + X^2 - t^2}{2xX}\right) H_0^{(1)}(k[t^2 + (y - Y)^2]^{1/2})\, dt \end{aligned} \tag{4.9}$$

and

$$G(x, y; X, Y) = \frac{-1}{4i} \int_{x-X}^{x-X} P_m \left(\frac{x^2 + X^2 - t^2}{2xX}\right) H_{0t}^{(1)}(k[t^2 + (y - Y)^2]^{1/2})\, dt. \tag{4.10}$$

So, for any positive integer m, we have shown that the addition formula yields the convergent integrals (4.9) and (4.10) which also satisfy (4.1) for $(x, y) \neq (X, Y)$. If we now differentiate the $H_0^{(1)}$ term as indicated in (4.10) and then perform the change of variables

$$s = [t^2 + (y - Y)^2]^{1/2}$$

we get

$$G(x, y; X, Y) = \frac{1}{4i} \int_r^{\bar{r}} P_m \left(\frac{x^2 + X^2 + (y - Y)^2 - s^2}{2xX}\right) H_1^{(1)}(ks)\, k\, ds. \tag{4.11}$$

In this form, we shall show that G has the appropriate singularity, $G \to 0$ as $x \to 0$, and $G \to 0$ as $r \to \infty$.

For notational convenience in investigating the behavior of G near the singular point $r = 0$, let us write

$$\zeta = \frac{x^2 + X^2 + (y - Y)^2 - s^2}{2xX} \equiv \frac{\bar{r}^2 + r^2 - 2s^2}{\bar{r}^2 - r^2}, \tag{4.12}$$

the identity following directly from the definitions of r and $\tilde{r}$. From (4.12) it is evident that, for $0 < r \leqslant s$ and for $s \to 0$, $\zeta \to 1$. Therefore

$$\lim_{0<r\leqslant s\to 0} P_m(\zeta) = P_m(1) = 1$$

by the continuity of all the functions involved. So given $\epsilon > 0$ arbitrarily small, choose $\delta > 0$ such that

$$0 < r \leqslant s \leqslant \delta \Rightarrow 1 - \epsilon \leqslant P_m(\zeta) < 1 \tag{4.13}$$

and such that

$$Y_1(ks) < 0$$

is simultaneously satisfied, where of course Y_1 is the first order Bessel function of the second kind (i.e., $Y_1 = \operatorname{Im}\{H_1^{(1)}\}$).

Now for $0 < r < \delta$ rewrite (4.11) as

$$G = \frac{1}{4i}\int_r^{\tilde{r}} P_m(\zeta)\, J_1(ks)\, k\, ds + \frac{1}{4}\int_r^{\delta} P_m(\zeta)\, Y_1(ks)\, k\, ds + \frac{1}{4}\int_\delta^{\tilde{r}} P_m(\zeta)\, Y_1(ks)\, k\, ds. \tag{4.14}$$

The first and third integrals in (4.14) are well-behaved and approach constant values as $r \to 0$. Recalling (4.13), we can apply the mean value theorem to the remaining integral resulting in

$$G = \tfrac{1}{4}P_m(\zeta^*)\,[Y_0(kr) - Y_0(k\delta)] + \text{constant}$$

where ζ^* is evaluated at $s^* \in (r, \delta)$. We know that for any $s \in (0, \delta)$, $1 - \epsilon \leqslant P_m \leqslant 1$ so that

$$\tfrac{1}{4}(1 - \epsilon)\, Y_0(kr) + \text{constant} \leqslant G \leqslant \tfrac{1}{4} Y_0(kr) + \text{constant}.$$

Thus, as $r \to 0$, G behaves like $\frac{1}{4}Y_0(kr)$ or equivalently $1/2\pi \log r$ [Ref. 7, pg. 360, 9.8.1], as desired. We have now verified conditions (1), (2), and (3) of section (3c) and all that remains is an investigation of the boundary conditions.

Approaching the singular line $x = 0$ in the half-plane $x > 0$, we have $0 < r < \tilde{r}$. In this case, for $r \leqslant s \leqslant \tilde{r}$, as the limits of integration in (4.11) indicate, we observe from (4.12) that $-1 \leqslant \zeta \leqslant 1$ and therefore $| P_m(\zeta)| \leqslant 1$ [Ref. 8]. We can now write

$$\lim_{x\to 0} | G | \leqslant \lim_{x\to 0} \tfrac{1}{4} \int_r^{\tilde{r}} | P_m(\zeta)| \cdot | H_1^{(1)}(ks)|\, k\, ds$$

$$\leqslant \lim_{x\to 0} \tfrac{1}{4} \int_r^{\tilde{r}} | H_1^{(1)}(ks)|\, k\, ds = 0$$

since as $x \to 0$, $r \to \tilde{r}$ and r is strictly positive (as a matter of fact $r \geqslant X > 0$ on $x = 0$) and thus $| H_1^{(1)}(ks)|$ is a continuous bounded function in the interval from r to $\tilde{r}$.

Lastly, we investigate the behavior of G as $r \to \infty$. Once again, $0 < r < \tilde{r}$ and $| P_m(\zeta)| \leqslant 1$ so that, from (4.11),

$$\begin{aligned} | G | &\leqslant \tfrac{1}{4} \int_r^{\tilde{r}} | H_1^{(1)}(ks)| \, k \, ds \\ &\leqslant \frac{k}{4} \, | H_1^{(1)}(kr)| \, (\tilde{r} - r) \end{aligned} \tag{4.15}$$

since $| H_1^{(1)}(kr)|$ is a monotonically decreasing function. [Ref. 6, pg. 969, 8.478]. From the definitions of r and $\tilde{r}$ we find that

$$0 < \tilde{r} - r = r\left[\left(1 + \frac{4xX}{r^2}\right)^{1/2} - 1\right].$$

For r sufficiently large, simple manipulations show that

$$0 \leqslant \tilde{r} - r < 2X + O\left(\frac{1}{r}\right). \tag{4.16}$$

For r large, it is also known [Ref. 7, pg. 365, 9.2.28] that

$$| H_1^{(1)}(kr)| = O(r^{-1/2}). \tag{4.17}$$

Substituting (4.16) and (4.17) into (4.15) yields

$$| G | \leqslant O(r^{-1/2})$$

and thus $G \to 0$ as $r \to \infty$. Hence, we have demonstrated that (4.11) satisfies (4.2), (4.2a), and (4.2b).

For the case $m = 1$, (4.11) becomes easy to integrate, utilizing the facts that $P_1(\zeta) = \zeta$ and

$$\int s^2 H_1^{(1)}(ks) \, k \, ds = s^2 H_2^{(1)}(ks) = \frac{2s}{k} H_1^{(1)}(ks) - s^2 H_0^{(1)}(ks) \tag{4.18}$$

[see Ref. 6, pg. 634, 5.52(1) and pg. 967, 9.471].

From the first equality in (4.18) we integrate (4.11) to

$$\begin{aligned} G(x, y; X, Y) &= \frac{1}{4i} \frac{x^2 + X^2 + (y - Y)^2}{2xX} [H_0^{(1)}(kr) - H_0^{(1)}(k\tilde{r})] \\ &\quad + \frac{1}{4i} \frac{1}{2xX} [r^2 H_2^{(1)}(kr) - \tilde{r}^2 H_2^{(1)}(k\tilde{r})] \end{aligned} \tag{4.19}$$

$$\begin{aligned} &= \frac{1}{4i} \frac{\tilde{r}^2 + r^2}{\tilde{r}^2 - r^2} [H_0^{(1)}(kr) - H_0^{(1)}(k\tilde{r})] \\ &\quad + \frac{1}{4i} \frac{2}{\tilde{r}^2 - r^2} [r^2 H_2^{(1)}(kr) - \tilde{r}^2 H_2^{(1)}(k\tilde{r})]. \end{aligned} \tag{4.20}$$

Utilizing the second form of (4.18) yields

$$G(x, y; X, Y) = \frac{1}{4i} H_0^{(1)}(kr) + \frac{1}{4i} H_0^{(1)}(k\tilde{r}) - \frac{i}{k(\tilde{r}^2 - r^2)} [rH_1^{(1)}(kr) - \tilde{r}H_1^{(1)}(k\tilde{r})]. \tag{4.21}$$

Using standard ascending series forms for $J_n(z)$ and $Y_n(z)$ [Ref. 7, pg. 360, 9[Ref. 7, pg. 360, 9.1.10 and 9.1.11] it is easy to show that

$$H_0^{(1)}(z) = \frac{2i}{\pi} \log z + \frac{2i}{\pi} \left[\frac{\pi}{2i} + \gamma - \log 2\right] + O(z^2 \log z)$$

and

$$H_2^{(1)}(z) = \frac{4}{\pi i} \frac{1}{z^2} + \frac{1}{\pi i} + O(z \log z).$$

Substituting these forms into (4.19) and letting $k \to 0$ yields

$$G(x, y; X, Y) = \frac{1}{2\pi} \left(\frac{x^2 + X^2 + (y - Y)^2}{2xX}\right) \log \frac{r}{\tilde{r}} + \frac{1}{2\pi}. \tag{4.22}$$

Thus (4.22) is the Green's function for

$$u_{xx} + u_{yy} - \frac{2}{x^2} u = 0 \tag{4.23}$$

satisfying the boundary conditions $G \to 0$ as $x \to 0$ and as $|(x - X, y - Y)| \to \infty$. It is interesting to note that (4.22) displays precisely the same form as a fundamental solution [Ref. 9].

If we allow k to approach 0 through positive values in (4.11) we get

$$G(x, y; X, Y) = -\frac{1}{2\pi} \int_r^{\tilde{r}} P_m \left(\frac{x^2 + X^2 + (y - Y)^2 - s^2}{2xX}\right) \frac{ds}{s} \tag{4.24}$$

(4.24) can also be derived directly from the addition formula (4.8) where G_1 still is the Green's function for (4.3), but $G_2 = 1/2\pi \log r$ is the Green's function for Laplace's equation (3.36). In the case $m = 1$, (4.24) integrates directly to our solution (4.22). Since P_m is simply a polynomial, a straightforward integration of (4.24) may be performed for all positive integers m, yielding closed form Green's function for

$$u_{xx} + u_{yy} - \frac{m(m + 1)}{x^2} u = 0$$

which satisfy the boundary conditions $G \to 0$ as $x \to 0$ and as $r \to \infty$.

5. Summary

In section 4, we applied the addition formula to the equation

$$u_{xx} + u_{yy} + \left[k^2 - \frac{m(m+1)}{x^2}\right] u = 0 \tag{5.1}$$

and verified that the integral expression we derived was indeed a Green's function. For the particular case $m = 1$, we were able to express the solution in a closed form. In addition, we succeeded in letting $k \to 0$ in the general integral form yielding Green's functions for (5.1) with $k = 0$. While no claims have been made about the uniqueness of our solutions, we feel that, for these equations the presentation of solutions, in such relatively simple forms, possessing the correct singularity and homogeneous boundary conditions is, in and of itself, of great interest and importance.

Furthermore, the addition formula derived in section 3 yields a new method of searching for Green's functions for separable elliptic partial differential equations. Of course, some analysis must be performed for each problem encountered, but our successful application of the formula to the equations studied in sections 3 and 4 encourages us to attempt to apply the formula to other cases.

Appendix I: Proof of (4.4)

Consider

$$L[G] = G_{xx} - G_{yy} - \frac{m(m+1)}{x^2} G = \delta(x - X)\,\delta(y - Y). \tag{I.1}$$

Multiply both sides of (I.1) by $x^{1/2} J_{m+1/2}(\lambda x)$ and integrate with respect to x from 0 to ∞. After some manipulation (including two integrations by parts and an application of Bessel's ordinary differential equation)

$$g_{yy} + \lambda^2 g = -X^{1/2} J_{m+1/2}(\lambda X)\,\delta(y - Y) \tag{I.2}$$

where

$$g(y; X, Y, \lambda) = \int_0^\infty x^{1/2} J_{m+1/2}(\lambda x)\, G(x, y; X, Y)\, dx. \tag{I.3}$$

Recalling our previous discussions on the behavior of Green's functions for hyperbolic equations and the similar form of (3.11), the solution of (I.2) is

$$g(y - Y; X, \lambda) = -X^{1/2} J_{m+1/2}(\lambda X)\, H(y - Y) \frac{\sin \lambda(y - Y)}{\lambda}. \tag{I.4}$$

Noting from (I.3) that g is the Fourier–Bessel transform of $x^{-1/2}G$, we can inverse transform (I.4) and multiply by $x^{1/2}$ to yield

$$G(x, y; X, Y) = -(xX)^{1/2}\, H(y - Y) \int_0^\infty \sin \lambda(y - Y)\, J_{m+1/2}(\lambda x)\, J_{m+1/2}(\lambda X)\, d\lambda.$$

Reference to an appropriate table of integrals [Ref. 6, pg. 732, 6.672 (1)] yields the desired result, which is Eq. (4.4).

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